Assessment of the modulated gradient model in decaying isotropic turbulence

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A recently introduced nonlinear model underwent evaluation based on two isotropic turbulent cases: a WISC case at a moderate Reynolds number and a JHU case at a high Reynolds number. The model uses an estimation of the subgrid-scale (SGS) kinetic energy to model the magnitude of the SGS stress tensor, and uses the normalized velocity gradient tensor to model the structure of the SGS stress tensor. Testing was performed for the first case through a comparison between direct numerical simulation (DNS) results and large eddy simulation results regarding resolved kinetic energy and the energy spectrum. In the second case, we examined resolved kinetic energy, the energy spectrum, as well as other key statistics including the probability density functions of velocities and velocity gradients, the skewness factor, and the flatness factor. Simulations using the model were numerically stable, and results were satisfactorily compared with DNS results and consistent with statistical theories of turbulence.
Keywords: large-eddy simulation, subgrid-scale model, isotropic turbulence

Large-eddy simulation (LES) is a tool for the study of high-Reynolds-number turbulent flows. The physical basis for LES is the separation of the flow into resolved and subgrid-scale (SGS) motions. The resolved motions contain most of the energy and can be computed numerically by solving LES governing equations, while effects of the less energetic SGS motions are parameterized.

It has come to our attention that a recently introduced nonlinear model can achieve stable simulations of high-Reynolds-number atmospheric boundary layer turbulent flows, and can deliver the expected logarithmic velocity profile in the near-wall region, the correct spectral scaling, and some other key statistics.[1] This letter presents an assessment of this model in decaying isotropic turbulent flows in order to more comprehensively understand its characteristics and ensure its capabilities as a simple alternative to eddy-viscosity model.

The LES governing equations of homogeneous incompressible isotropic flow are

$$\frac{\partial \widetilde{u}_i}{\partial x_i} = 0 \ , \ \frac{\partial \widetilde{u}_i}{\partial t} + \frac{\partial \widetilde{u}_i \widetilde{u}_j}{\partial x_j} = -\frac{\partial \widetilde{p}}{\partial x_i} + \nu \frac{\partial^2 \widetilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j} \ , \quad (1)$$

where $\widetilde{p}$ is the effective pressure, ν is the kinematic viscosity, and the SGS stress tensor is $\tau_{ij} = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j$, which must be closed in terms of the resolved field.

Numerous SGS models have been proposed since the 1960s. Eddy-viscosity models are the most commonly used SGS models in LES. Based on the Boussinesq hypothesis,[2] they are constructed using the constitutive relations

$$\tau_{ij} - \frac{\tau_{kk}}{3}\delta_{ij} = -2\nu_t \widetilde{S}_{ij} \ , \quad (2)$$

where $\widetilde{S}_{ij} = \left(\partial \widetilde{u}_i/\partial x_j + \partial \widetilde{u}_j/\partial x_i\right)/2$, and ν_t is the eddy viscosity. The Smagorinsky model (SM)[3] assumes that the eddy viscosity is modeled as $\nu_t = (C_s \widetilde{\Delta})^2 |\widetilde{S}|$, where $|\widetilde{S}| = \left(2\widetilde{S}_{ij}\widetilde{S}_{ij}\right)^{1/2}$ is the strain rate, and we adopt the Smagorinsky coefficient, $C_s = 0.17$, in the study as suggested in previous studies.[4]

The variety of SGS models arises not only because the theoretical justifications are arguable but also because LES solutions are sensitive to the type of SGS model. Different from eddy-viscosity models, the gradient model (GM, also referred to as a "nonlinear model" or "Clark model") derives from the Taylor series expansions of the SGS terms that appear in the LES equations [5, 6]

$$\tau_{ij} = G_{ij} \ , \quad (3)$$

where $\widetilde{G}_{ij} = \frac{\widetilde{\Delta}^2}{12}\left(\frac{\partial \widetilde{u}_i}{\partial x_k}\frac{\partial \widetilde{u}_j}{\partial x_k}\right)$, and makes no use of prior knowledge of the interactions between the resolved motions and the SGS motions. At *a-priori* level, the GM predicts the structure of the exact SGS terms much more accurate than that eddy-viscosity models do (and then are better able to capture anisotropic effects and disequilibrium [7–9]). These features make the GM attractive.

However, when implemented in LESs, the GM performs less efficiently for dealing with the level of energy dissipation, as a result, simulations often become numerically unstable as reported.[10]. This has led to so-called mixed models, in which an $O(\widetilde{\Delta}^2)$ or $O(\widetilde{\Delta}^4)$ viscosity term can be added.[9, 11] As an alternative approach to resolving this issue, the modulated gradient model (MGM) has been introduced [1]

$$\tau_{ij} = 2k_{sgs}\left(\frac{\widetilde{G}_{ij}}{\widetilde{G}_{kk}}\right) \ . \quad (4)$$

The MGM uses an estimation of the SGS kinetic energy ($k_{sgs} = \frac{1}{2}\tau_{ii}$) to model the magnitude of the SGS stress tensor, and the normalized velocity gradient tensor to model the structure of the SGS stress tensor, i.e, the relative magnitude of the different tensor components. To evaluate k_{sgs}, we adopt the "local" equilibrium hypothesis, which assumes a balance between SGS energy production and dissipation rate. SGS energy production is defined as $P = -\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} = -\tau_{ij}\widetilde{S}_{ij}$. A simple evaluation of kinetic energy dissipation is $\varepsilon = C_\varepsilon \frac{k_{sgs}^{3/2}}{\widetilde{\Delta}}$, and the coefficient is assumed to be $C_\varepsilon = 1$ based on previous studies.[12, 13] For ensuring numerical stability, no local energy transfer from unresolved to resolved scales is allowed, a step consistent with the fact that dissipation rate is nonnegative, then one can obtain

$$k_{sgs} = H\left(P\right)\frac{4\widetilde{\Delta}^2}{C_\varepsilon^2}\left(-\frac{\widetilde{G}_{ij}}{\widetilde{G}_{kk}}\widetilde{S}_{ij}\right)^2, \tag{5}$$

where $H(x)$ is the Heaviside step function defined as $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$.

In this model assessment study, we adopt two decaying isotropic turbulent cases. Decaying isotropic turbulence is governed by two key elements, nonlinearity and viscosity; thus is a typical setup for the assessment of SGS models. All simulations are performed on periodic grids in a $[0, 2\pi]^3$ domain using a modified version of the pseudo-spectral code used for previous studies.[8, 9, 14, 15] The viscous term is solved using an integrating factor, which helps increase numerical stability and to decrease numerical diffusion. The time advance is carried out through the use of an explicit third order Runge-Kutta scheme. The corresponding aliasing errors are corrected in the nonlinear terms according to the 2/3 rule in a cubic-truncation manner.[16] In LES, the filter size is implicit, and previous research has suggested that the ratio of the filter size to the grid size is in a range of $1 \sim 2$.[10] We have found that the filter size, $\widetilde{\Delta} = 1.5h$ (where h is the grid size), yields satisfactory results. Also, we use a small Courant-Friedrichs-Lewy number of 0.1 to suppress numerical dissipations. Note that in the presentation, we normalize the time scale using the initial eddy turn-over time.

A direct numerical simulation (DNS) case has been simulated at the University of Wisconsin-Madison (WISC) by means of decaying, and has been adopted in previous model assessment studies.[8, 9] The initial Taylor micro-scale Reynolds number (Re_λ) is approximately 85. Thus, the 128^3 DNS has resolved the flow of all scales, and the DNS results can be used to verify the accuracy of SGS models and identify their problems. Figure 1(a) shows the evolution of the resolved kinetic energy obtained from DNS and LESs. The decaying case starts at $Re_\lambda = 85$, and is beyond the capability of 64^3 simulation (typically $Re_\lambda \sim 50$). The total resolved kinetic energy obtained from the 64^3 simulation without a model, which simply omits τ_{ij}, yields the worst prediction. In order to obtain proper kinetic energy decay rates, turbulence modeling is needed for coarser grids. It is clear that the results obtained using the MGM are in the best agreement with the filtered DNS results.

The SM yields a higher kinetic energy decay rate than that is found in the filtered DNS results. Figure 1(b) shows that using the SM, kinetic energy at small scales is dissipated excessively. By construction, the GM allows for energy "backscatter." However, over a period of time,

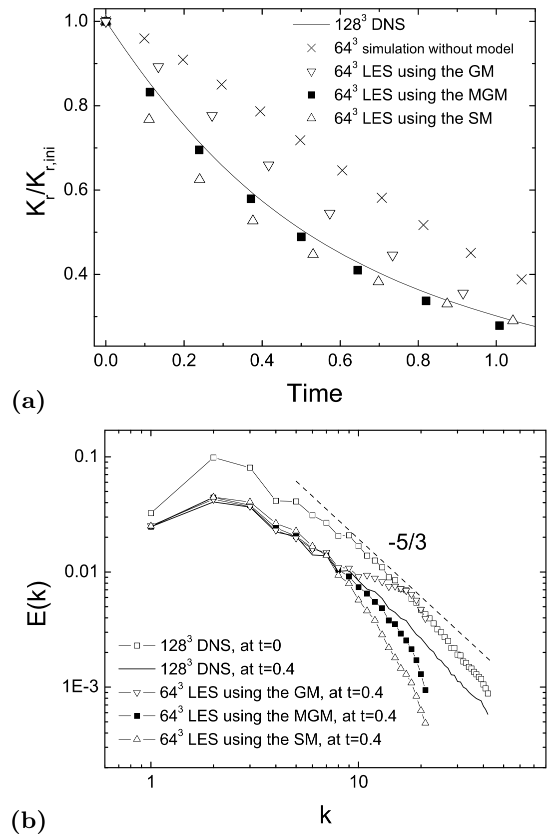

FIG. 1. (a) Evolution of resolved kinetic energy (normalized by its initial value); (b) comparison of energy spectra at $t = 0.4$.

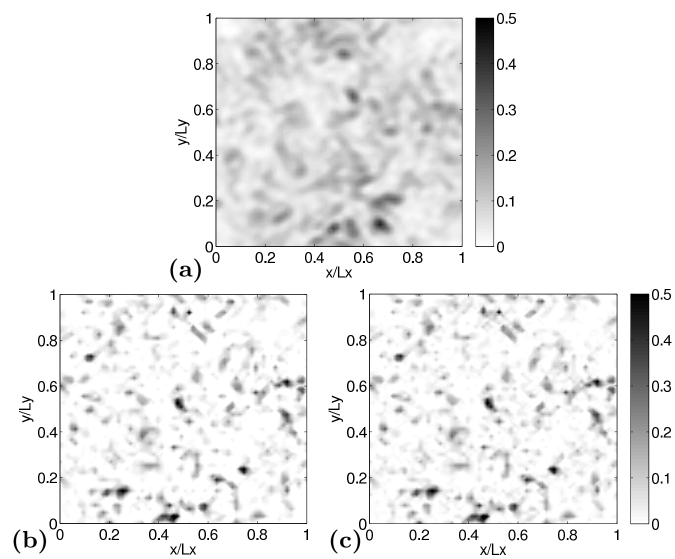

FIG. 2. Initial instantaneous contours of: (a) exact k_{sgs} obtained from DNS; (b) k_{sgs} obtained using the clipping procedure; and (c) k_{sgs} obtained without clipping.

the GM yields a lower kinetic energy decay rate, and kinetic energy at small scales is accumulating (cannot be dissipated effectively) as shown in figure 1(b). As its revision, the MGM has shown significant improvement in energy-spectrum accuracy.

Figure 2 compares the SGS kinetic energy obtained from the filtered DNS data, and two evaluations using the clipping procedure (Eqn. 5) and without clipping (the step function is disabled). The estimated SGS kinetic energy follows the same order as the exact SGS kinetic energy. However, the discrepancy in distributions is noticeable. Alternative ways of computing k_{sgs} that might improve accuracy, include solving an additional transportation equation. A comparison between figure 2(b) and figure 2(c) removes any worry that the clipping procedure, by possibly yielding high-level jumps, would thus create numerical instabilities. Clearly, high-magnitude k_{sgs} contours are identical, and the clipping procedure merely smooths out any low-magnitude k_{sgs}.

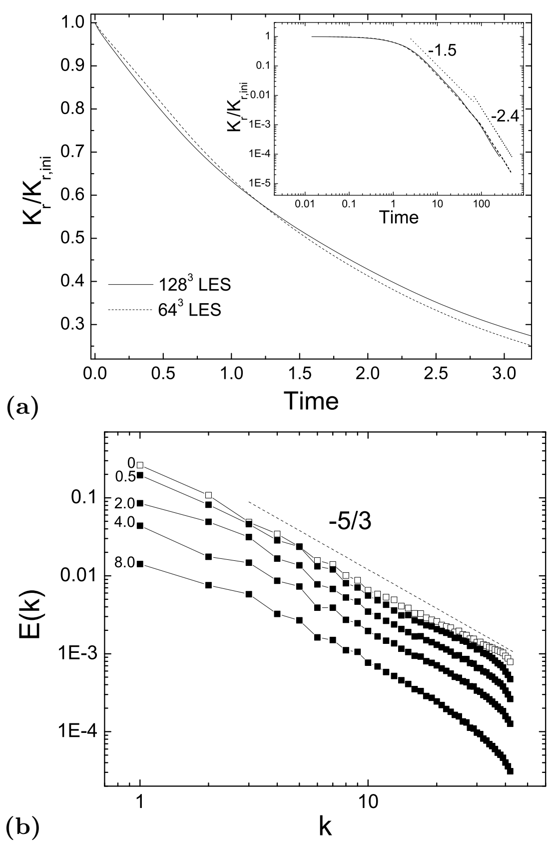

FIG. 3. (a) Evolution of resolved kinetic energy (normalized by its initial value); (b) energy spectra obtained from 128^3 LES at $time = 0$, 0.5, 2.0, 4.0 and 8.0.

For further testing of the new model, a high-Reynolds-number case is conducted. The original simulation was performed at the Johns Hopkins University (JHU) using 1024 grid points in each direction.[17] The database contains a 1024^4 space-time history of an incompressible isotropic turbulent flow in 3D. The initial condition for decaying LES runs, downloaded from "turbulence.pha.jhu.edu" at time equals 2 without space interpolation, bears $Re_\lambda = 430$. A direct numerical simulation of decay was lacking; thus comparisons could be performed only against statistical theories of turbulence. Figure 3(a) shows the evolution of the resolved kinetic energy obtained from the LESs using the new model at the resolutions of 128^3 and 64^3. Overall, two simulations are in good agreement. The dissipate rate of the 64^3 LES is slightly lower in the early period. The top-right corner presents their power-law decay behaviors. Previous experiments, DNSs and analyses have found that two decay exponents, one during the initial period and the other during the final period are depending on the initial conditions.[18, 19] Self-similar solution shows that the shape of the kinetic-energy spectrum at a low wavenumber determines the initial decay rate, and the highest decay exponents are 2 during the initial decay and 2.5 during the final period.[19] In the current study, simulations yield the initial and final decay exponents are approximately 1.5 and 2.4. Figure 3(b) shows the kinetic energy spectra at different times. They follow the $-5/3$ power-law behavior until the dissipation range starts to be captured through LES at a late period, and importantly, there are no improper accumulations of kinetic energy at small scales.

Figures 4(a) and 4(b) show the probability density functions (PDFs) of velocities and longitudinal velocity gradients, and figure 4(c) shows the evolutions of skewness and flatness factors. The velocity derivatives are calculated from the Fourier components of the velocity field and then transformed into physical space. The results are in good agreement with the results reported in the literature.[17, 19–21] Theoretically, PDFs of turbulent velocities are near-Gaussian; thus the skewness factors of velocities are almost zero and the flatness factors of velocities are close to 3. Whereas, velocity derivatives are not exactly Gaussian random variables as shown that the skewness factors of velocity gradients have a negative value of approximately -0.35 and the flatness factors of velocity gradients are close to 3.8. The lack of Gaussianity is an intrinsic feature of turbulence because of the nonlinearity of the Navier-Stokes equations, and the new model can achieve this feature.

In summary, we have tested a recently introduced nonlinear SGS model in simulations of two decaying isotropic turbulent cases. The model uses the normalized velocity gradient tensor to model the structure of the SGS stress tensor, and estimates the SGS kinetic energy on the basis of the "local" equilibrium hypothesis. There is no requirement of an extra filtering - thus, the model is computationally efficient. Applications to decaying isotropic turbulent cases at a moderate Reynolds number and at a high Reynolds number show that the model can achieve reasonable spectral scaling, as well as some key statistical

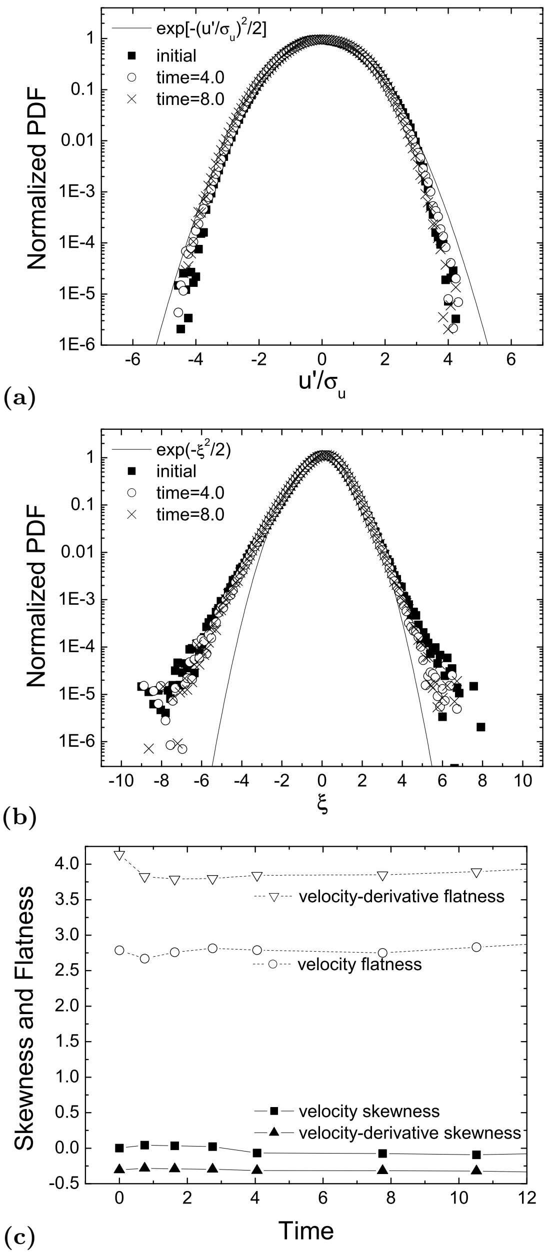

FIG. 4. (a) Probability density functions (PDFs) of velocities; (b) PDFs of longitudinal velocity gradients; and (c) evolutions of skewness and flatness factors. PDFs are normalized by the multiplication of $\sqrt{2\pi}$ and rms values; and variables are normalized by rms values. The solid line is Gaussian distribution.

characteristics of isotropic turbulence.

In its present formulation, the model needs *a-priori* knowledge to determine the model coefficient. The selected constant value is based on theoretical arguments, which are strictly valid only in the inertial subrange of high-Reynolds-number turbulence. Possible future modifications of the model include the development and testing of dynamic and scale-dependent dynamic procedures to optimize the value of the model coefficient using information of the resolved velocity field. Moreover, when computing the SGS kinetic energy, researchers seeking to improve accuracy could consider alternative approaches, including solving an additional transportation equation.

The author is indebted to Fernando Porté-Agel, Leslie Smith, Charles Meneveau and Hyung-Suk Kang for stimulation and discussion. This research was supported by the National Science Foundation (grants EAR-0537856 and ATM-0854766), NASA (grant NNG06GE256), customers of Xcel Energy through a grant (RD3-42) from the Renewable Development Fund, and the University of Minnesota Institute for Renewable Energy and the Environment. Computing resources were provided by the Minnesota Supercomputing Institute.

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